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The Spectral Gap of the 2-D Stochastic Ising Model with Mixed Boundary Conditions

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We establish upper bounds for the spectral gap of the stochastic Ising model at low temperatures in an $l \times l$ box with boundary conditions which are not purely plus or minus; specifically, we assume the magnitude of the sum of the boundary spins over each interval of length l in the boundary is bounded by δl, where $\delta < 1$. We show that for any such boundary condition, when the temperature is sufficiently low (depending on δ), the spectral gap decreases exponentially in l.

KEY WORDS: Stochastic Ising model; spectral gap; Glauber dynamics.

1. INTRODUCTION

1.1. General Background and Heuristics

We begin with an informal description; full definitions will be given below. Consider the stochastic Ising model (Glauber dynamics) in an $l \times l$ box $\Lambda(l)$, below the critical temperature. At equilibrium, the typical configuration has one or more macroscopic regions each resembling one of the two infinite-volume pure phases (plus phase or minus phase) except very near the boundary; these regions are arranged so as to minimize the surface energy of any interfaces between them. Thus in nearly every small subregion, the equilibrium distribution $\mu = \mu^{\beta}_{\Lambda(l), \omega}$ (at inverse temperature β, under boundary condition ω) is roughly either the plus phase, the minus

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phase or a distributional mixture of the two. This equilibrium may take a long time to be reached, if the box is large. The rate of convergence is described by the spectral gap, denoted $\text{gap}(\Lambda(l), \omega, \beta)$, which is the smallest positive eigenvalue of the negative of the generator of the dynamics. More precisely, for $S(\cdot)$ the associated semigroup and $\|\cdot\|_\mu$ the $L^2(\mu)$ norm, $\text{gap}(\Lambda(l), \omega, \beta)$ is the largest constant Δ such that

$$\left\| S(t)\, f - \int f\, d\mu \right\|_\mu \leqslant \left\| f - \int f\, d\mu \right\|_\mu e^{-\Delta t} \qquad \text{for all} \quad f \in L^2(\mu) \text{ and } t \geqslant 0.$$

For pure boundary conditions, say all plus, at subcritical temperatures the spectral gap is believed to be of order l^{-2} [FH87]. The spectral gap can be very sensitive to the boundary condition, however. For example, removing as few as $O(\log l)$ plus spins near each corner of $\Lambda(l)$ (leaving the boudary there free, or minus) yields a gap much smaller than l^{-2}, and removing ϵl plus spins from each corner, for some positive ϵ, yields a gap which decreases exponentially in l [Al00]. These phenomena are outgrowths of the fact that the boundary conditions are not well mixed, the free boundary or minus spins being concentrated in short intervals at the corners. More mixed boundary conditions are considered in [HY97], where it is shown that if the boundary condition ω satisfies

$$\left| \sum_{y \in I} \omega_y \right| \leqslant \delta l/2 \qquad \text{for every interval } I \text{ in } \partial_{\text{ex}} \Lambda(l) \tag{1.1}$$

with $\delta < 1$, then

$$\text{gap}(\Lambda(l), \omega, \beta) \leqslant B_{1.2} \exp(-\beta l / C_{1.2}), \qquad l = 1, 2, \ldots, \tag{1.2}$$

where $B_{1.2} = B_{1.2}(\beta) > 0$ and $C_{1.2} > 0$. Here $\partial_{\text{ex}} \Lambda(l)$ denotes the exterior boundary; see (1.8). One can allow the boundary spins ω_y to take values in the continuum $[-1, 1]$, with $\omega_y = 0$ representing the free boundary condition at site y. The condition (1.1) is somewhat restrictive, however; for example, it does not allow the long intervals of boundary plus spins which appear in the above-mentioned results from [Al00]. In this paper we establish (1.2) under a "mixed boundary" hypothesis much weaker than (1.1).

The importance of the geometry of boundary spin locations can be seen in comparing the result in [Al00], giving exponential decay of the gap when ϵl plus spins are removed at each corner, to a result of Martinelli [Mar94] which states that when one side of the square has all-plus boundary condition, and the other 3 sides have free boundary, at sufficiently low temperatures one has

$$\exp(-C(\beta, \epsilon)\, l^{\frac{1}{2}+\epsilon}) \leqslant \text{gap}(\Lambda(l), \omega, \beta) \qquad \text{for} \quad \epsilon > 0, \quad l = 1, 2,.... \tag{1.3}$$

In the latter case there are many fewer plus spins but the gap is much larger, meaning the convergence to the equilibrium plus phase is much faster.

The heuristics of the gap are rooted in the ideas of energy barriers and traps. From certain starting configurations, to reach a typical equilibrium configuration, one must pass through a set of configurations for each of which the total energy is greater than either the typical starting or equilibrium total energies. An *energy barrier* is such a set of high-energy configurations; the *height* of the barrier is the typical additional energy of the barrier configurations relative to the starting configurations. A *trap* is a set of starting configurations from which one cannot reach equilibrium without crossing an energy barrier. (We do not make formal definitions here, as we will not use these concepts other than descriptively.) Typically one expects the gap to be exponentially small in the height of the energy barrier that must be crossed, for a trap of which the probability is "not too small." Often traps are related to the existence of macroscopic regions of the "wrong phase," that is, say, regions of minus phase when the equilibrium is purely the plus phase. For example, in the "corners-removed" context of [Al00], a trap is formed by the configurations in which there is an "X" of minus phase connecting the four free-boundary corner regions, and the height of the associated energy barrier is proportional to the length of the corner regions. In the above "three-sides-free" example of Martinelli, however, say with the plus spins on the right side of the square, there is no real energy barrier because, starting from the minus phase, a region of plus phase can sweep leftward, maintaining an approximately vertical interface, until it covers the full square.

Consider now a boundary condition ω which is "well-mixed" in the sense that

$$\left| \sum_{y \in I} \omega_y \right| \leqslant \delta\, |I| \tag{1.4}$$

for every "sufficiently long" interval I in the boundary of $\Lambda(l)$, with $\delta < 1$, and suppose ω favors the plus phase (more precisely, the magnetization at the center of the square is nonnegative.) If the system is started entirely in the minus phase, we expect the region of minus phase (the "droplet") to pull away from at least one side of the boundary and then shrink to nothing, at which time equilibrium is essentially reached. When the droplet initially fills $\Lambda(l)$, the energy associated to its surface (this surface being essentially $\partial_{\text{ex}}\Lambda(l)$) is at most $8\delta l$, by (1.4). When one side of the droplet has pulled only slightly away from the boundary, however, the surface energy of that side becomes essentially twice its length (provided the temperature is very low), hence the surface energy of the whole droplet is at least about $(6\delta+2)\,l$. Thus there is an energy barrier; the droplet will tend to stick to the boundary, meaning the minus phase is a trap. Though we do not make these particular heuristics rigorous in our proofs, they are what underlie our main result.

For fixed ω satisfying (1.4), at higher but still subcritical temperatures, one does not expect this phenomenon of sticking to the boundary to occur. This is because the surface energy (appropriately defined using surface tension and coarse-graining) of the droplet is no longer essentially twice its length; a diagonal interface has significantly less surface energy than combined horizontal and vertical interfaces having the same endpoints. This means the droplet should be able to pull away from the boundary, first from the corners, without the crossing of an energy barrier. We will not investigate this type of behavior here.

Additional existing results at subcritical temperatures include the following. Thomas [Tho89] proved that in general dimension d, for free boundary conditions ($\omega\equiv 0$), for sufficiently large β,

$$\text{gap}(\Lambda(l),\omega,\beta)\leqslant B_{1.5}\exp(-\beta l^{d-1}/C_{1.5})\qquad l=1,2,..., \tag{1.5}$$

where $B_{1.5}=B_{1.5}(\beta,d)>0$ and $C_{1.5}=C_{1.5}(d)>0$. For $d=2$, Cesi *et al.* [CGMS96] prove (1.5) with $\omega\equiv 0$ for all $\beta>\beta_c$, where β_c is the inverse critical temparature. For $d=2$, in contrast with (1.5), it is known that for $\beta>\beta_c$ and $\omega\equiv +1$,

$$\exp(-\varphi(l))\leqslant \text{gap}(\Lambda(l),\omega,\beta),\qquad l=1,2,..., \tag{1.6}$$

with a function $\varphi(l)=o(l^{\frac{1}{2}+\epsilon})$ as $l\nearrow\infty$, for all $\epsilon>0$. This result was first obtained by F. Martinelli [Mar94, Mar97]. More recently, Y. Higuchi and J. Wang [HW99] showed (1.6) with $\varphi(l)=C(\beta)(l\ln l)^{\frac{1}{2}}$. Schonmann [Sch94, Theorem 5] showed that $\text{gap}(\Lambda(l),\omega)$ can shrink no faster than exponential of $O(l^{d-1})$; specifically, the spectral gap has the following general lower bound for all $d\geqslant 2$ and $\beta>0$:

$$\underline{q}(\beta)\,l^{-d}\exp\left(-4\beta\sum_{j=0}^{d-1}l^j\right)\leqslant \inf_{\omega\in\Omega_b}\text{gap}(\Lambda(l),\omega,\beta),\qquad l=1,2,.... \tag{1.7}$$

Here $\underline{q}(\beta)$ is a uniform lower bound for all flip rates.

1.2. Basic Definitions

The Lattice. For $x=(x_1, x_2) \in \mathbf{Z}^2$, we will use both the l_1-norm $\|x\|_1 = |x_1|+|x_2|$ and the l_∞-norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. A set $\Lambda \subset \mathbf{Z}^2$ is said to be *ℓ_p-connected* ($p=1$ or ∞) if for each distinct $x, y \in \Lambda$, we can find some $\{x_0,..., x_n\} \subset \Lambda$ with $x_0 = x$, $x_n = y$ and $\|x_j - x_{j-1}\|_p = 1$ ($j=1,..., n$). The exterior boundary of a set $\Lambda \subset \mathbf{Z}^2$ will be denoted by

$$\partial_{\mathrm{ex}} \Lambda = \{y \notin \Lambda;\ \|x-y\|_1 = 1 \text{ for some } x \in \Lambda\}. \tag{1.8}$$

The number of points contained in a set $\Lambda \subset \mathbf{Z}^2$ will be denoted by $|\Lambda|$. We will use the notation $\Lambda \subset\subset \mathbf{Z}^2$ to indicate that $\Lambda \subset \mathbf{Z}^2$ with $|\Lambda| < \infty$. A cube with the side-length l will be denoted by

$$\Lambda(l) = (-l/2, l/2]^d \cap \mathbf{Z}^d. \tag{1.9}$$

An ℓ_∞-connected subset of $\partial_{\mathrm{ex}} \Lambda(l)$ will be called an *interval* of $\partial_{\mathrm{ex}} \Lambda(l)$.

The Configurations and the Gibbs States. We define two kind of spin configurations;

$$\Omega_\Lambda = \{\sigma = (\sigma_x)_{x \in \Lambda};\ \sigma_x = +1 \text{ or } -1\}, \qquad \Lambda \subset\subset \mathbf{Z}^2,$$
$$\Omega_{\mathrm{b}} = \{\omega = (\omega_y)_{y \in \mathbf{Z}^2};\ \omega_y \in [-1, 1]\}.$$

We are mainly interested in $\omega_y \in \{-1, 0, 1\}$, but there is no extra work in allowing $\omega_y \in [-1, 1]$. We will refer an element ω of Ω_{b} as a boundary condition. The set of all real functions on Ω_Λ is denoted by $\mathscr{C}_\Lambda$. For $\Lambda \subset\subset \mathbf{Z}^2$ and $\omega \in \Omega_{\mathrm{b}}$, the *Hamiltonian* $H^\omega_\Lambda \colon \Omega_\Lambda \to \mathbf{R}$ is defined by

$$H^\omega_\Lambda(\sigma) = -\tfrac{1}{2} \sum_{\substack{x,\, y \in \Lambda \\ \|x-y\|_1 = 1}} \sigma_x \sigma_y - \sum_{\substack{x \in \Lambda,\, y \notin \Lambda \\ \|x-y\|_1 = 1}} \sigma_x \omega_y.$$

We let $\beta > 0$ denote the inverse temperature, and let $\mu^\beta_{\Lambda,\omega}$ denote the corresponding finite-volume Gibbs state.

Stochastic Ising Models. For $\Lambda \subset\subset \mathbf{Z}^2$ and β fixed, we consider the stochastic Ising model on Λ, with the flip rate at x in configuration σ under boundary condition ω denoted $q_\Lambda(x, \sigma, \omega)$. As is usual we assume *boundedness*, meaning there exist positive constants $\underline{q}(\beta)$ and $\bar{q}(\beta)$ such that

$$\underline{q}(\beta) \leqslant q_\Lambda(x, \sigma, \omega) \leqslant \bar{q}(\beta), \tag{1.10}$$

for all $\Lambda \subset\subset \mathbf{Z}^2$ and $(x, \sigma, \omega) \in \Lambda \times \Omega_\Lambda \times \Omega_{\mathrm{b}}$, and we assume the *detailed balance condition,* also known as reversibility (see [Li85].) Now, fix

$\Lambda \subset\subset \mathbf{Z}^2$ and $\omega \in \Omega_{\mathrm{b}}$. The generator of the process is the linear operator $A_\Lambda^\omega \colon \mathscr{C}_\Lambda \to \mathscr{C}_\Lambda$ given by

$$A_\Lambda^\omega f(\sigma) = \sum_{x \in \Lambda} q_\Lambda(x, \sigma, \omega)\{f(\sigma^x) - f(\sigma)\}, \qquad f \in \mathscr{C}_\Lambda.$$

As is well known, it follows from the detailed balance condition that

$$-\mu_{\Lambda,\omega}^\beta(f A_\Lambda^\omega g) = \tfrac{1}{2} \sum_{x \in \Lambda} \sum_{\sigma \in \Omega} \mu_{\Lambda,\omega}^\beta(\sigma) q_\Lambda(x, \sigma, \omega)\{f(\sigma^x) - f(\sigma)\}\{g(\sigma^x) - g(\sigma)\}. \tag{1.11}$$

Next, we define

$$\mathrm{gap}(\Lambda, \omega, \beta) = \inf\left\{ \frac{-\mu_{\Lambda,\omega}^\beta(f A_\Lambda^\omega f)}{\mu_{\Lambda,\omega}^\beta(|f - \mu_{\Lambda,\omega}^\beta f|^2)}\, ; f \in \mathscr{C}_\Lambda \right\}, \tag{1.12}$$

which is the smallest positive eigenvalue of $-A_\Lambda^\omega$. Considering only indicator functions in (1.12) we obtain

$$\mathrm{gap}(\Lambda, \omega, \beta) \leqslant \frac{\bar{q}(\beta)}{\mu_{\Lambda(l),\omega}^\beta(\Gamma)\, \mu_{\Lambda(l),\omega}^\beta(\Gamma^c)} \sum_{x \in \Lambda(l)} \sum_{\sigma \in \Gamma : \sigma^x \notin \Gamma} \mu_{\Lambda(l),\omega}^\beta(\sigma). \tag{1.13}$$

Thus any fixed event Γ gives an upper bound for the gap. Roughly, to obtain a good bound one wants to choose Γ to be a trap.

1.3. Statement of Main Results

The following is our main result, improving on the condition (1.1).

Theorem 1.1. Consider a stochastic Ising model on a square $\Lambda(l)$ for which the flip rates satisfy boundedness and the detailed balance condition. Suppose that $0 < \delta < 1$ and the boundary condition $\omega_y \in [-1, +1]$, $y \in \partial_{\mathrm{ex}} \Lambda(l)$ satisfies

$$\left| \sum_{y \in I} \omega_y \right| \leqslant \delta\, |I| \qquad \text{for every interval} \quad I \subset \partial_{\mathrm{ex}} \Lambda(l) \quad \text{with } |I| = l. \tag{1.14}$$

Then, there exists $\beta_0 = \beta_0(\delta) > 0$ such that

$$\mathrm{gap}(\Lambda(l), \omega, \beta) \leqslant B_{1.15} \exp(-\beta l / C_{1.14}), \tag{1.15}$$

for $\beta \geqslant \beta_0$ and $l = 1, 2, \ldots$, where $B_{1.15} = B_{1.15}(\beta, \delta) > 0$ and $C_{1.15} = C_{1.15}(\delta) > 0$.

Condition (1.14) is much milder than (1.1). For example, (1.14) allows a boundary condition which is $+1$ for 99 % of the boundary with 1 % zero on each side. Moreover, condition (1.14) turns out to be optimal in the following example. For $\delta > 0$, consider a boundary condition $\omega \in \Omega_b$ defined by

$$\omega_x = \begin{cases} +1 & \text{if } \ x_1 = [l/2]+1 \ \text{ and } \ \dfrac{-\delta l}{2} < x_2 \leqslant \dfrac{\delta l}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

In this example, we see the transition from (1.2) to (1.3) depending on the value of δ. By Theorem 1.1, one sees that (1.2) is true for all $\delta < 1$. On the other hand, it follows from [Mar94, Corollary 4.1] that (1.3) holds true for $\delta = 1$.

Theorem 1.1 has the following application to random boundary conditions.

Corollary 1.2. Suppose that $d = 2$ and that $\omega_y \in [-1, 1]$, $y \in \mathbf{Z}^2$ are i.i.d. random variables with the mean $m \in (-1, 1)$. Then, there exists $\beta_0 = \beta_0(m) > 0$ as follows. For $\beta \geqslant \beta_0$, there are constants $B_{1.17} = B_{1.17}(\beta, m) > 0$ and $C_{1.17} = C_{1.17}(m) > 0$ such that almost surely;

$$\text{gap}(\Lambda(l), \omega, \beta) \leqslant B_{1.17} \exp(-\beta l / C_{1.17}) \qquad \text{for} \quad l = 1, 2,... \tag{1.17}$$

Proof of Corollary 1.2 is similar to that of [HY97, Corollary 2.2.] and hence is omitted.

2. PRELIMINARIES FOR THE PROOF OF THEOREM 1.1

2.1. Contours

The set $\mathbf{B}$ of all bonds in $\mathbf{Z}^2$ is defined by

$$\mathbf{B} = \{\{x, y\} \subset \mathbf{Z}^2; \|x - y\|_1 = 1\}.$$

For a set Λ, we define

$$\begin{aligned} \mathbf{B}_\Lambda &= \{\{x, y\} \in \mathbf{B}; (x, y) \in \Lambda^2\}, \\ \partial \mathbf{B}_\Lambda &= \{\{x, y\} \in \mathbf{B}; (x, y) \in \Lambda \times \Lambda^c\}, \\ \overline{\mathbf{B}}_\Lambda &= \mathbf{B}_\Lambda \cup \partial \mathbf{B}_\Lambda. \end{aligned}$$

The *dual lattice* $(\mathbf{Z}^2)^*$ is $\mathbf{Z}^2$ shifted by $(\frac{1}{2}, \frac{1}{2})$; sites and bonds of this lattice are called *dual sites* and *dual bonds*. x^* denotes $x + (\frac{1}{2}, \frac{1}{2})$. When necessary for clarity, bonds of $\mathbf{Z}^2$ are called *regular bonds*. To each regular bond b there is associated a unique dual bond b^* which is its perpendicular bisector. For $A \subset \mathbf{B}$ we write A^* for $\{e^*: e \in A\}$. For $\gamma \subset \overline{\mathbf{B}}^*_\Lambda$ we set

$$V(\gamma) = \bigcup_{e = \{x, y\}:\, e^* \in \gamma} \{x, y\}, \qquad V_{\text{ex}}(\gamma) = V(\gamma) \cap \partial_{\text{ex}} \Lambda.$$

When convenient we view bonds and dual bonds as closed intervals in $\mathbf{R}^2$, as when referrring to a connected set of (dual) bonds. The number of dual bonds contained in a set $\gamma \subset \mathbf{B}^*$ will be denoted by $|\gamma|$.

For $x \in \mathbf{R}^2$ let $Q(x) = \prod_{j=1}^2 [x_j - \frac{1}{2}, x_j + \frac{1}{2}]$, and for $\Theta \subset \mathbf{Z}^2$ let $Q(\Theta) = \bigcup_{x \in \Theta} Q(x)$. A *contour* is a finite subset $\gamma \subset \mathbf{B}^*$ which is of the form $\partial Q(\Theta)$ for some finite $\Theta \subset \mathbf{Z}^2$ for which both Θ and Θ^c are l_1-connected. The set Θ is uniquely determined by γ and hence is denoted by $\Theta(\gamma)$. As is well known, for each $b \in \mathbf{B}$ and $m = 1, 2, \ldots$,

$$\#\{\gamma: \gamma \text{ is a contour with } |\gamma| = m \text{ and } \gamma \notin b\} \leqslant 3^{m-1}. \tag{2.1}$$

If a contour γ is a subset of $\overline{\mathbf{B}}^*_\Lambda$ for some $\Lambda \subset \mathbf{Z}^2$, we say γ is a contour in Λ. For $\sigma \in \Omega_\Lambda$, $\varepsilon = +$ or $-$ and $\Lambda \subset \mathbf{Z}^2$, an *$(\epsilon)$-cluster* in Λ at σ is an l_1-connected component of $\{x \in \mathbf{Z}^2 : \sigma_x = \epsilon 1\}$. The *outer boundary* of a bounded subset A of $\mathbf{R}^2$ is the unique connected component of ∂A which is contained in the closure of the unique unbounded component of A^c. A contour γ is said to be an (ε)-contour in Λ at σ if γ is the outer boundary of $Q(\Theta)$ for some (ε)-cluster Θ. A contour γ is said to be a contour in Λ at $\sigma \in \Omega_\Lambda$ if it is either $(+)$-contour in Λ at σ or $(-)$-contour in Λ at σ. Note that the boundary condition does not affect whether a given γ is an (ε)-contour in Λ, under our definition.

2.2. Outline of the Proof of Theorem 1.1

It is easy to check that (1.13) implies that there exists $0 < \delta_{2.2} < 1$ such that

$$\left|\sum_{y \in I} \omega_y\right| \leqslant \delta_{2.2}\, |I| \qquad \text{for every interval} \quad I \subset \partial_{\text{ex}} \Lambda(l) \quad \text{with } |I| \geqslant \delta_{2.2} l, \tag{2.2}$$

so we henceforth assume (2.2).

The basic strategy to prove Theorem 1.1 is rather standard [Tho89, HY97]. We define an event $\Gamma_l \subset \Omega_{\Lambda(l)}$ in which a "large" contour is present, and apply (1.12) with $\Gamma = \Gamma_l$. It is thus enough to show that for large β, $\mu^{\beta}_{\Lambda(l),\omega}(\Gamma_l)$ is uniformly positive in l (Lemma 2.1 below) and

$$\mu^{\beta}_{\Lambda(l),\,\omega}(\Gamma^c_l)^{-1} \sum_{x \in \Lambda(l)} \sum_{\sigma \in \Gamma_l,\, \sigma^x \notin \Gamma_l} \mu^{\beta}_{\Lambda(l),\,\omega}(\sigma)$$

is exponentially small in l (Lemma 2.2 below). We may assume that

$$\mu^{\beta}_{\Lambda(l),\,\omega}(\sigma_0) \geqslant 0. \tag{2.3}$$

We fix δ_1 such that $\delta_{2.2} < \delta_1 < 1$. The event Γ_l is defined by

$$\Gamma_l = \{\sigma \in \Omega_{\Lambda(l)};\, C_l(\sigma) \neq \emptyset\}, \tag{2.4}$$

where

$$C_l(\sigma) = \left\{ \gamma; \begin{array}{c} \gamma \text{ is a } (+)\text{-contour in } \Lambda(l) \text{ at } \sigma \text{ such that} \\ \gamma \cap \partial Q(\Lambda(l)) \neq \emptyset \text{ and } |\gamma| \geqslant 2\delta_1 l \end{array} \right\}. \tag{2.5}$$

Thus we must prove the following results.

Lemma 2.1. Assume (2.2) and (2.3). There exists $\beta_1 = \beta_1(\delta_{2.2}) > 0$ such that

$$\inf_{l \geqslant 1} \mu^{\beta}_{\Lambda(l),\,\omega}(\Gamma_l) \geqslant \tfrac{1}{3} \qquad \text{for} \quad \beta \geqslant \beta_1. \tag{2.6}$$

Lemma 2.2. Assume (2.2) and (2.3). There exists $\beta_2 = \beta_2(\delta_{2.2}) > 0$ such that

$$\sum_{x \in \Lambda(l)} \sum_{\sigma \in \Gamma_l : \sigma^x \notin \Gamma_l} \mu^{\beta}_{\Lambda(l),\,\omega}(\sigma) \leqslant \mu^{\beta}_{\Lambda(l),\,\omega}(\Gamma^c_l)\, B_{2.7} \exp(-\beta l / C_{2.7}) \tag{2.7}$$

for $\beta \geqslant \beta_2$ and $l = 1, 2, \ldots,$ where $B_{2.7} = B_{2.7}(\beta, \delta_{2.2}) > 0$ and $C_{2.7} = C_{2.7}(\delta_{2.2}) > 0$.

Theorem 1.1 follows immediately by plugging (2.6) and (2.7) into (1.12). In fact, we have for $\beta \geqslant \max\{\beta_1, \beta_2\}$ that

$$\operatorname{gap}(\Lambda(l), \omega, \beta) \leqslant 3\bar{q}(\beta)\, B_{2.7} \exp(-\beta l / C_{2.7}). \tag{2.8}$$

3. PROOF OF LEMMAS 2.1 AND 2.2

3.1. Energy Estimates for Contours

The proofs of Lemmas 2.1 and 2.2 are based on energy estimates for contours, which we present in this subsection. We have to introduce additional definitions. The right, left, top and bottom sides of the square $Q(\Lambda(l))$ are denoted by $F_l^{+1}, F_l^{-1}, F_l^{+2}$ and F_l^{-2}, respectively. A set of dual bonds $\gamma \subset \overline{\mathbf{B}}_\Lambda^*$ is said to be *horizontally crossing* if γ intersects both F_l^{-1} and F_l^1; *vertically crossing* is defined analogously. The set γ is said to be *crossing* if it is either horizontally crossing or vertically crossing.

Suppose that $\gamma_1,...,\gamma_p$ are contours in $\Lambda(l)$. We set

$$\Delta_{\gamma_1,..., \gamma_p} H^\omega_{\Lambda(l)}(\sigma) = H^\omega_{\Lambda(l)}(\sigma) - H^\omega_{\Lambda(l)}(T_{\gamma_1} \circ \cdots \circ T_{\gamma_p}\sigma), \qquad \sigma \in \Omega_{\Lambda(l)} \tag{3.1}$$

where we have defined a map $T_\gamma\colon \Omega_{\Lambda(l)} \to \Omega_{\Lambda(l)}$ for a contour γ by

$$(T_\gamma \sigma)_x = \begin{cases} -\sigma_x, & \text{if} \quad x \in \Theta(\gamma) \\ \sigma_x, & \text{if} \quad x \notin \Theta(\gamma). \end{cases} \tag{3.2}$$

Suppose that a contour γ is non-crossing. Then $\gamma \cap (F_l^i \cup F_l^j) = \varnothing$ for some i, j with $|i| = 1$ and $|j| = 2$. Then there exists a unique connected component $\underline{\gamma}$ of $\gamma \setminus \partial Q(\Lambda(l))$ which divides $\Lambda(l)$ into two l_1-connected components $\widetilde{\Theta}$ and $\Lambda(l) \setminus \widetilde{\Theta}$ such that $\Theta(\gamma) \subset \widetilde{\Theta}$ and $F_l^i \cup F_l^j \subset \partial Q(\Lambda(l) \setminus \widetilde{\Theta})$. We define $\bar{\gamma} \subset \partial Q(\Lambda(l))$ and the interval $I(\gamma) \subset \partial_{\text{ex}} \Lambda(l)$ respectively by

$$\bar{\gamma} = \partial Q(\Lambda(l)) \cap \partial Q(\widetilde{\Theta}), \qquad I(\gamma) = V_{\text{ex}}(\bar{\gamma}). \tag{3.3}$$

Note that

$$\bar{\gamma} \supset \gamma \cap \partial Q(\Lambda(l)). \tag{3.4}$$

Note also that bonds in $\bar{\gamma}$ are in one-to-one correspondence with sites in $I(\gamma)$ in an obvious way.

Lemma 3.1.

(a) Let γ be a a non-crossing (ϵ)-contour at a configuration $\sigma \in \Omega_{\Lambda(l)}$. Then

$$|\gamma \setminus \partial Q(\Lambda(l))| \geqslant \tfrac{1}{2}|\gamma| + \tfrac{1}{2}|\gamma \setminus (\partial Q(\Lambda(l)) \cup \underline{\gamma})|, \tag{3.5}$$

$$\tfrac{1}{2}\Delta_\gamma H^\omega_{\Lambda(l)}(\sigma) \geqslant \tfrac{1}{2}|\gamma| + \tfrac{1}{2}|\gamma \setminus (\partial Q(\Lambda(l)) \cup \underline{\gamma})| - \epsilon \sum_{y \in V_{\mathrm{ex}}(\gamma)} \omega_y, \tag{3.6}$$

$$\tfrac{1}{2}\Delta_\gamma H^\omega_{\Lambda(l)}(\sigma) \geqslant |\underline{\gamma}| - \epsilon \sum_{y \in V_{\mathrm{ex}}(\bar{\gamma})} \omega_y \geqslant 0. \tag{3.7}$$

(b) Suppose $\{\gamma_j\}_{j=1}^p$ are (ϵ)-contours in $\Lambda(l)$ at σ such that

$$\tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^\omega_{\Lambda(l)}(\sigma) \geqslant c_1 l - c_2 \qquad \text{for some} \quad c_i \geqslant 0 \quad (i = 1, 2). \tag{3.8}$$

Then,

$$\frac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^\omega_{\Lambda(l)}(\sigma) \geqslant \frac{c_1}{c_1 + 8} \sum_{j=1}^p |\gamma_j| - c_2. \tag{3.9}$$

(c) Let γ, $\gamma_1,\ldots, \gamma_p$ be non-crossing (ϵ)-contours at a configuration $\sigma \in \Omega_{\Lambda(l)}$. Suppose that condition (2.2) is satisfied and that I is an interval in $\partial_{\mathrm{ex}} \Lambda(l)$ such that

$$\bigcup_{j=1}^p I(\gamma_j) \subset I, \tag{3.10}$$

$$\delta_{2.2} l \leqslant |I| \leqslant \sum_{j=1}^p |\underline{\gamma_j}| + c, \tag{3.11}$$

where $c \geqslant 0$ is a constant. Then

$$\tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^\omega_{\Lambda(l)}(\sigma) \geqslant \varepsilon_{3.12} \max\left\{l, \sum_{j=1}^p |\gamma_j|\right\} - c, \tag{3.12}$$

where the constant $\varepsilon_{3.12} > 0$ depends only on $\delta_{2.2}$.

Proof. (a) is a straightforward exercise which we omit. To prove (b), let $\alpha = 1/(c_1 + 8)$.

Case 1: $\alpha \sum_{j=1}^{p} |\gamma_j| \leqslant l$. In this case, we obviously have that

$$\tfrac{1}{2}\Delta_{\gamma_1, \ldots, \gamma_p} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant c_1 \alpha \sum_{j=1}^{p} |\gamma_j| - c_2.$$

Case 2: $\alpha \sum_{j=1}^{p} |\gamma_j| \geqslant l$. In this case,

$$\begin{aligned}
\tfrac{1}{2}\Delta_{\gamma_1, \ldots, \gamma_p} H^{\omega}_{\Lambda(l)}(\sigma) &\geqslant \sum_{j=1}^{p} (|\gamma_j \setminus \partial Q(\Lambda(l))| - |\gamma_j \cap \partial Q(\Lambda(l))|) \\
&= \sum_{j=1}^{p} (|\gamma_j| - 2\, |\gamma_j \cap \partial Q(\Lambda(l))|) \\
&\geqslant \sum_{j=1}^{p} |\gamma_j| - 8l \\
&\geqslant (1 - 8\alpha) \sum_{j=1}^{p} |\gamma_j|.
\end{aligned}$$

Therefore (3.9) follows.

For (c), it is enough to prove (3.8) with some $c_1 > 0$ and $c_2 = c$. Recall that $\delta_{2.2} < \delta_1 < 1$.

Case 1: $\sum_{j=1}^{p} |I(\gamma_j)| \leqslant \delta_1\, |I|$. In this case,

$$\begin{aligned}
\tfrac{1}{2}\Delta_{\gamma_1, \ldots, \gamma_p} H^{\omega}_{\Lambda(l)}(\sigma) &\geqslant \sum_{j=1}^{p} (|\underline{\gamma_j}| - |I(\gamma_j)|) \\
&\geqslant |I| - \sum_{j=1}^{p} |I(\gamma_j)| - c \\
&\geqslant (1 - \delta_1)\, |I| - c \\
&\geqslant (1 - \delta_1)\, \delta_{2.2} l - c,
\end{aligned}$$

which implies (3.8) with $c_1 = (1 - \delta_1)\, \delta_{2.2}$.

Case 2: $\sum_{j=1}^{p} |I(\gamma_j)| \geqslant \delta_1\, |I|$. We set $A = I \setminus \bigcup_{j=1}^{p} I(\gamma_j)$ so that $|A| \leqslant (1 - \delta_1)\, |I|$. We then have by (3.7), (3.11), (2.2) that

$$\begin{aligned}\tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma) &\geqslant \sum_{j=1}^{p} \Big(|\underline{\gamma_j}| - \epsilon \sum_{y \in I(\gamma_j)} \omega_y\Big) \\ &\geqslant |I| - c - \epsilon \sum_{y \in I} \omega_y - |A| \\ &\geqslant |I| - c - \delta_{2.2}\, |I| - (1-\delta_1)\, |I| \\ &\geqslant (\delta_1 - \delta_{2.2})\, \delta_{2.2} l - c,\end{aligned}$$

which implies (3.8) with $c_1 = (\delta_1 - \delta_{2.2})\, \delta_{2.2}$. ▮

Lemma 3.2. Let γ be an (ϵ)- contour in $\Lambda(l)$ at a configuration σ.

(a) If γ intersects with exactly one of the sides F^j_l ($j = \pm 1, \pm 2$), then

$$\Delta_\gamma H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \begin{cases} |\{\text{horizontal bonds in } \gamma\}| & \text{if} \quad j = \pm 1, \\ |\{\text{vertical bonds in } \gamma\}| & \text{if} \quad j = \pm 2. \end{cases} \tag{3.13}$$

(b) If $\Theta(\gamma) \notin 0$ and $|\gamma| < 2l$, then

$$\Delta_\gamma H^{\omega}_{\Lambda(l)}(\sigma) \geqslant 2|\gamma|/9. \tag{3.14}$$

Proof. Part (a) is straightforward. Part (b) follows readily from (a) and the fact that when $\Theta(\gamma) \notin 0$ and $|\gamma| < 2l$, γ intersects at most one of the sides F^l_j. ▮

3.2. Proof of Lemmas 2.1 and 2.2

The proof of Lemma 2.1 is a standard Peierls argument, so we omit it. For Lemma 2.2, we proceed as follows.

Step 1: Suppose that $\sigma \in \Gamma_l$ and $\sigma^x \notin \Gamma_l$ for some $x \in \Lambda(l)$. We consider two cases separately at first: $\sigma_x = 1$ and $\sigma_x = -1$.

Consider first $\sigma_x = 1$. Let γ be the outer boundary of the $(+)$-cluster at σ which contains x. The way the transition from $\sigma \in \Gamma_l$ to $\sigma^x \notin \Gamma_l$ occurs is that the set $C_l(\sigma)$ contains only the one element γ, and the flipping of σ_x shortens γ or separates γ from $\partial Q(\Lambda(l))$ or makes γ break into new shorter contours. Some of these shorter contours may include dual bonds which were not part of γ at σ, but rather were part of $(-)$-contours inside γ at σ. We have then

$$C_l(\sigma) = \{\gamma\}, \tag{3.15}$$

$$x \text{ is in or adjacent to } V(\gamma); \tag{3.16}$$

in fact if either (3.15) or (3.16) fails, then $C_l(\sigma) = C_l(\sigma^x)$, contradicting our assumption that $\sigma \in \Gamma_l$ and $\sigma^x \notin \Gamma_l$. Further, there are $(+)$-contours $\gamma_1,..., \gamma_m$ and $\gamma'_1,..., \gamma'_n$ ($m \geqslant 0$, $n \geqslant 0$, $1 \leqslant m+n \leqslant 4$) at the flipped configuration σ^x, and $(-)$-contours $\alpha_1,..., \alpha_k$ ($0 \leqslant k \leqslant 1$) inside γ at σ, such that

$$\gamma_j \cap \partial Q(\Lambda(l)) \neq \varnothing, \qquad |\gamma_j| < 2\delta_1 l, \quad \text{for } j = 1, ..., m, \tag{3.17}$$

$$\gamma'_j \cap \partial Q(\Lambda(l)) = \varnothing, \qquad \text{for } j = 1, ..., n, \tag{3.18}$$

$$\left(\gamma \cup \left(\bigcup_{j=1}^{k} \alpha_j\right)\right) \triangle \left(\left(\bigcup_{j=1}^{m} \gamma_j\right) \cup \left(\bigcup_{j=1}^{n} \gamma'_j\right)\right) \subset \partial Q(x), \tag{3.19}$$

$$\Theta(\gamma) \triangle \left(\left(\bigcup_{j=1}^{m} \Theta(\gamma_j)\right) \cup \left(\bigcup_{j=1}^{n} \Theta(\gamma'_j)\right) \cup \left(\bigcup_{j=1}^{k} \Theta(\alpha_j)\right)\right) = \{x\}, \tag{3.20}$$

where $\triangle$ stands for the symmetric difference of two sets. Each α_j, γ_j and γ'_j must surround at least one neighbor of x. $\gamma_1,..., \gamma_m$ and $\gamma'_1,..., \gamma'_n$ are precisely the $(+)$-contours at σ_x which include bonds of γ. Let

$$S_- = \sum_{j=1}^{m} |\gamma_j| + \sum_{j=1}^{n} |\gamma'_j|, \qquad S_+ = |\gamma| + \sum_{j=1}^{k} |\alpha_j|.$$

Using (3.19) it is easy to see that

$$S_+ \leqslant S_- \leqslant S_+ + 4. \tag{3.21}$$

We will show that

$$\Delta_{\gamma, \alpha_1,..., \alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \varepsilon_{3.22} S_+ - C_{3.22}, \tag{3.22}$$

where $\varepsilon_{3.22} = \varepsilon_{3.22}(\delta) > 0$ and $C_{3.22} = C_{3.22}(\delta) > 0$, by using (3.19) and studying the contours $\gamma_1,..., \gamma_m$ and $\gamma'_1,..., \gamma'_n$.

Now consider $\sigma_x = -1$. In this case, one possibility is that the flipping of σ_x connects together two or three $(+)$-clusters to create a $(+)$-cluster which has a shorter outer boundary than the longest of the original $(+)$-clusters had. If we let γ denote the outer boundary of the $(+)$-cluster of x at σ^x, this means there are again $(+)$-contours $\gamma_1,..., \gamma_m$ and $\gamma'_1,..., \gamma'_n$ ($m \geqslant 0$, $n \geqslant 0$, $1 \leqslant m+n \leqslant 2$), and $(-)$-contours $\alpha_1,..., \alpha_k$ ($0 \leqslant k \leqslant 2$) inside γ, such that (3.18)–(3.20) hold, but now γ and the α_j exist at σ^x and the γ_j and γ'_j exist at σ. Further, $|\gamma| < 2\delta_1 l$, and in place of (3.17),

$$\gamma_j \cap \partial Q(\Lambda(l)) \neq \varnothing, \quad \text{for } j = 1,..., m, \qquad |\gamma_j| \geqslant 2\delta_1 l \quad \text{for some } j. \tag{3.23}$$

(The other possiblity when $\sigma_x = -1$ is that only one $(+)$-cluster (call it C) at σ is contained in the $(+)$-cluster of x at σ^x, and the flipping of σ_x to 1 shortens the boundary of C; this may be taken as another case of the above with $m+n=1$ and $0 \leqslant k \leqslant 3$.) Here in place of (3.15) we have

$$C_l(\sigma) = \{\gamma_j : 1 \leqslant j \leqslant m, |\gamma_j| \geqslant 2\delta_1 l\}. \tag{3.24}$$

Statement (3.16) still holds, and in place of (3.22) we will prove

$$\Delta_{\gamma_1,\ldots,\gamma_m,\gamma'_1,\ldots,\gamma'_n} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \varepsilon_{3.25} S_- - C_{3.25}, \tag{3.25}$$

It is easy to see that for fixed x, both for $\sigma_x = 1$ and for $\sigma_x = -1$, the sets $\{\gamma_1,\ldots,\gamma_m, \gamma'_1,\ldots,\gamma'_n\}$ and $\{\gamma, \alpha_1,\ldots,\alpha_k\}$ uniquely determine each other.

We now turn to the proof of (3.22) for $\sigma_x = 1$ and (3.25) for $\sigma_x = -1$. For $\sigma_x = 1$ we have using (3.20) that

$$\begin{aligned}\tfrac{1}{2}\Delta_{\gamma,\alpha_1,\ldots,\alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) &\geqslant \tfrac{1}{2}\Delta_{\gamma,\alpha_1,\ldots,\alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) - 4\\ &\geqslant \tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_m,\gamma'_1,\ldots,\gamma'_n} H^{\omega}_{\Lambda(l)}(\sigma^x) - 8\\ &= \tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_m} H^{\omega}_{\Lambda(l)}(\sigma^x) + \tfrac{1}{2}\Delta_{\gamma'_1,\ldots,\gamma'_n} H^{\omega}_{\Lambda(l)}(T\sigma^x) - 8\end{aligned} \tag{3.26}$$

where $T = T_{\gamma_1} \circ \cdots \circ T_{\gamma_m}$ (Recall (3.2)). Each contour in $\{\gamma_j\}$ is non-crossing, since $|\gamma_j| < 2\delta_1 l$. Therefore, we see from (3.6) that for any $0 \leqslant p \leqslant m$,

$$\begin{aligned}\tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_m} H^{\omega}_{\Lambda(l)}(\sigma^x) &\geqslant \tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma^x) + \sum_{j=p+1}^{m} \left(\tfrac{1}{2}|\gamma_j| - |\gamma_j \cap \partial\Lambda(l)|\right)\\ &\geqslant \tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma^x)\\ &\geqslant 0.\end{aligned} \tag{3.27}$$

On the other hand, we have

$$\Delta_{\gamma'_1,\ldots,\gamma'_n} H^{\omega}_{\Lambda(l)}(T\sigma^x) = 2\sum_{j=1}^{n} |\gamma'_j|, \tag{3.28}$$

since $\gamma'_j \cap \partial\Lambda(l) = \varnothing$. We have as a consequence that

$$\tfrac{1}{2}\Delta_{\gamma,\alpha_1,\ldots,\alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma^x) + \sum_{j=1}^{n} |\gamma'_j| - 8. \tag{3.29}$$

Note also that the first term on the right-hand-side of (3.29) is non-negative by (3.27). For $\sigma_x = -1$, γ is non-crossing since $|\gamma| < 2\delta_1 l$, and each α_j is inside γ, so it follows from (3.19) that

$$\partial Q(x) \cup \left(\bigcup_{j=1}^{m} \gamma_j \right) \text{ is non-crossing in } Q(\Lambda(l)); \tag{3.30}$$

in particular each γ_j is non-crossing. Therefore (3.26)–(3.29) remain valid but with σ and σ^x interchanged; in fact we may replace (3.29) with

$$\tfrac{1}{2}\Delta_{\gamma_1,\dots,\gamma_m,\gamma'_1,\dots,\gamma'_n} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \tfrac{1}{2}\Delta_{\gamma_1,\dots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma) + \sum_{j=1}^{n} |\gamma'_j|. \tag{3.31}$$

To bound (3.29) or (3.31) from below, we pick a number δ_2 such that $\delta_{2.2} < \delta_2 < \delta_1$ and consider the following three cases separately.

Case 1: $S_+ \geqslant 9l$. Here the possible energy gain along $\partial\Lambda(l)$ when T is applied is not enough to cancel the energy reduction in the interior. Specifically, for $\sigma_x = 1$,

$$\begin{aligned} \tfrac{1}{2}\Delta_{\gamma_1,\dots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma^x) &\geqslant \sum_{j+1}^{m} |\gamma_j \setminus \partial Q(\Lambda(l))| - \sum_{y \in \partial\Lambda(l)} |\omega_y| \\ &\geqslant \sum_{j+1}^{m} |\gamma_j| - 8l. \end{aligned} \tag{3.32}$$

With (3.21) and (3.29) this shows

$$\tfrac{1}{2}\Delta_{\gamma,\alpha_1,\dots,\alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant S_+ - 8l - 8 \geqslant \tfrac{1}{9} S_+ - 8, \tag{3.33}$$

which proves (3.22). The same argument with (3.31) replacing (3.29) proves (3.25) when $\sigma_x = -1$.

Case 2: $\sum_{j=1}^{m} |\gamma_j| \leqslant (\delta_2/\delta_1)\, S_+$. Consider first $\sigma_x = 1$. Here by (3.21)

$$\sum_{j=1}^{n} |\gamma'_j| \geqslant S_+ - \sum_{j=1}^{m} |\gamma_j| \geqslant \left(1 - \frac{\delta_2}{\delta_1}\right) S_+, \tag{3.34}$$

which, together with (3.29), proves (3.22) in this case. Using again (3.21), the same argument with (3.31) replacing (3.29) proves (3.25) when $\sigma_x = -1$.

Case 3: $\sum_{j=1}^{m} |\gamma_j| > (\delta_2/\delta_1)\, S_+$ and $S_+ < 9l$. By (3.21), (3.29) and (3.31) it is enough to prove that

$$\Delta_{\gamma_1,\ldots,\gamma_p} H^{\omega}_{\Lambda(l)}(\sigma^x) \geqslant \varepsilon l - c \tag{3.35}$$

for some $\varepsilon > 0$, $c \geqslant 0$ and $1 \leqslant p \leqslant m$. We consider several subcases as follows.

Case 3.1: $\sum_{j=1}^{m} |\gamma_j \cap \partial Q(\Lambda(l))| \leqslant \delta_{2.2} l$. Consider first $\sigma_x = 1$. Since

$$\delta_2 l \leqslant \tfrac{1}{2}(\delta_2/\delta_1)\,|\gamma| \leqslant \tfrac{1}{2}(\delta_2/\delta_1)\, S_+ \leqslant \tfrac{1}{2}\sum_{j=1}^{m} |\gamma_j|, \tag{3.36}$$

we have by (3.27) that

$$\tfrac{1}{2}\Delta_{\gamma_1,\ldots,\gamma_m} H^{\omega}_{\Lambda(l)}(\sigma^x) \geqslant (\delta_2 - \delta_{2.2})\, l$$

which proves (3.35). For $\sigma_x = -1$, in place of (3.36) we use (3.23) to obtain

$$\delta_2 l \leqslant \delta_1 l \leqslant \tfrac{1}{2}\sum_{j=1}^{m} |\gamma_j|;$$

otherwise the argument for (3.35) is the same.

Case 3.2: The set $\partial Q(x) \cup (\bigcup_{j=1}^{m} \gamma_j)$ is non-crossing in $\Lambda(l)$ and

$$\sum_{j=1}^{m} |\gamma_j \cap \partial Q(\Lambda(l))| \geqslant \delta_{2.2} l. \tag{3.37}$$

In this case we have

$$\left(\bigcup_{j=1}^{m} \gamma_j\right) \cap (F^i_l \cup F^k_l) = \varnothing$$

for some i, k with $|i| = 1$ and $|k| = 2$. Then there exists a connected subset, say λ, of

$$\left(\bigcup_{j=1}^{m} \underline{\gamma_j}\right) \cup \partial Q(x)$$

which divides $\Lambda(l)$ into two connected components $\tilde{\Theta}$ and $\Lambda(l)\setminus\tilde{\Theta}$ such that $\bigcup_{j=1}^{m} \Theta(\gamma_j) \subset \tilde{\Theta}$ and $F^i_l \cup F^k_l \subset \partial_{\text{ex}}(\Lambda(l)\setminus\tilde{\Theta})$. Note that the set I defined by $I = \partial_{\mathrm{ex}} \Lambda(l) \cap \partial\widetilde{\Theta}$ is an interval. To prove (3.35) by applying (3.12), let us check (3.10) and (3.11) with $p = m$. We see from the construction of I that

$$\bigcup_{j=1}^{m} I(\gamma_j) \subset I,$$

$$|I| \leqslant |\lambda| \leqslant \sum_{j=1}^{m} |\underline{\gamma_j}| + 4.$$

On the other hand, we see from (3.37) that

$$|I| \geqslant \sum_{j=1}^{m} |\gamma_j \cap \partial\Lambda(l)| \geqslant \delta_{2.2} l.$$

We therefore have (3.10) and (3.11) with $p = m$.

Case 3.3: The set $\partial Q(x) \cup (\bigcup_{j=1}^{m} \gamma_j)$ is crossing in $\Lambda(l)$. By (3.30) this is possible only when $\sigma_x = 1$. There exist $1 \leqslant i < j \leqslant 4$ and $k \in \{1, 2\}$ such that

$$\gamma_i \cap F_l^k \neq \varnothing \qquad \text{and} \qquad \gamma_j \cap F_l^{-k} \neq \varnothing. \tag{3.38}$$

Let us assume (3.38) with $i = 1$, $j = 2$, and $k = 1$. Then, the set $\gamma_1 \cup \gamma_2$ cannot be vertically crossing, since $|\gamma_1| + |\gamma_2| < 4\delta_1 l$ and $\gamma_1 \cup \gamma_2$ is already horizontally crossing. Let us therefore assume that

$$(\gamma_1 \cup \gamma_2) \cap F_l^{-2} = \varnothing. \tag{3.39}$$

We are now left with two possibilities.

Case 3.3.1: $\gamma_1 \cap F_l^2 \neq \varnothing$ and $\gamma_2 \cap F_l^2 \neq \varnothing$. In this case, $I = I(\gamma_1) \cup I(\gamma_2) \cup F_l^2$ is an interval. To prove (3.35) by applying (3.12), we will check (3.10) and (3.11) with $p = 2$. We obviously have

$$\bigcup_{j=1}^{2} I(\gamma_j) \subset I,$$

$$|I| \geqslant l.$$

On the other hand, it is easy to see we have the following injections:

$$I(\gamma_1) \cap F_l^1 \to \{\text{vertical dual bonds in } \underline{\gamma_1}\},$$
$$I(\gamma_2) \cap F_l^{-1} \to \{\text{vertical dual bonds in } \underline{\gamma_2}\},$$
$$F_l^2 \to \{\text{horizontal dual bonds in } (\underline{\gamma_1} \cup \underline{\gamma_2}) \cup \partial Q(x)\}.$$

We get $|I| \leqslant |\underline{\gamma_1}| + |\underline{\gamma_2}| + 4$ as a consequence. We therefore have (3.10) and (3.11) with $p = 2$.

Case 3.3.2: $\gamma_1 \cap F_l^2 = \emptyset$ or $\gamma_2 \cap F_l^2 = \emptyset$. Let us assume $\gamma_1 \cap F_l^2 = \emptyset$, so that γ_1 does not intersect with F_l^j, $j \neq 1$. The distance from x to F_l^{-1} is at most $\delta_1 l$, and hence the distance from x to F_l^1 is at least $(1-\delta_1)\, l$. This implies that γ_1 deviates from F_l^1 at least by distance $(1-\delta_1)\, l - 1$. Therefore, by (3.13),

$$\tfrac{1}{2} \Delta_{\gamma_1} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant |\{\text{horizontal bonds in } \gamma_1\}|$$

$$\geqslant 2(1-\delta_1)\, l - 2,$$

which establishes (3.35) with $p = 1$.

Step 2: Suppose again that $\sigma \in \Gamma_l$ and $\sigma^x \notin \Gamma_l$. If $\sigma_x = 1$, then every $(+)$-contour outside γ at σ has length at most $2\delta_1 l$, and every $(+)$- or $(-)$-contour inside γ does not intersect $\partial Q(\Lambda(l))$. It follows that

$$T_\gamma \circ T_{\alpha_1} \circ \cdots \circ T_{\alpha_k} \sigma \in \Gamma_l^c. \tag{3.40}$$

Similarly if $\sigma_x = -1$ then

$$T_{\gamma_1} \circ \cdots \circ T_{\gamma_m} \circ T_{\gamma'_1} \circ \cdots \circ T_{\gamma'_n} \sigma \in \Gamma_l^c. \tag{3.41}$$

Step 3: For $x \in \Lambda(l)$ and $\sigma \in \Gamma_l$ with $\sigma^x \notin \Gamma_l$, let $\mathscr{C}_+(\sigma, x)$ denote the set of contours $\{\gamma, \alpha_1,..., \alpha_k\}$, defined previously, if $\sigma_x = 1$, and let $\mathscr{C}_-(\sigma, x) = \{\gamma_1,..., \gamma_m, \gamma'_1,..., \gamma'_n\}$ if $\sigma_x = -1$. We have by obervations made in Step 1 that

$$\sum_{x \in \Lambda(l)} \sum_{\sigma \in \Gamma_l : \sigma^x \notin \Gamma_l, \sigma_x = 1} \mu^{\beta}_{\Lambda(l), \omega}(\sigma)$$

$$\leqslant \sum_{x \in \Lambda(l)} \sum_{\gamma, \alpha_1,..., \alpha_k} \mu^{\beta}_{\Lambda(l), \omega}$$

$$\{\sigma: \mathscr{C}_+(\sigma, x) = \{\gamma, \alpha_1,..., \alpha_k\},\ \Delta_{\gamma, \alpha_1,..., \alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \varepsilon_{3.22} S_+ - C_{3.22}\} \tag{3.42}$$

where $\sum_{\gamma, \alpha_1,..., \alpha_k}$ stands for the summation over all possible values of $\mathscr{C}_+(\sigma, x)$. Now for fixed x and n there are at most $c \cdot 3^n$ possible values of $\mathscr{C}_+(\sigma, x)$ for which $S_+ = n$. By the standard Peierl's argument and the obervation made in Step 2, we can proceed as follows, provided β is sufficiently large:

$$\sum_{x \in \Lambda(l)} \sum_{\gamma, \alpha_1,\dots,\alpha_k} \mu^{\beta}_{\Lambda(l),\omega}\{\sigma\colon \mathscr{C}_+(\sigma,x) = \{\gamma, \alpha_1,\dots,\alpha_k\},$$
$$\Delta_{\gamma,\alpha_1,\dots,\alpha_k} H^{\omega}_{\Lambda(l)}(\sigma) \geqslant \varepsilon_{3.22} S_+ - C_{3.22}\}$$
$$\leqslant \sum_{x \in \Lambda(l)} \sum_{\gamma, \alpha_1,\dots,\alpha_k} \exp(-\beta(\varepsilon_{3.22} S_+ - C_{3.22}))$$
$$\cdot \mu^{\beta}_{\Lambda(l),\omega}\{T_\gamma \circ T_{\alpha_1} \circ \cdots \circ T_{\alpha_k}\sigma\colon \mathscr{C}_+(\sigma,x) = \{\gamma,\alpha_1,\dots,\alpha_k\}\}$$
$$\leqslant \sum_{x \in \Lambda(l)} \sum_{n \geqslant 2\delta_1 l} c \cdot 3^n \exp(-\beta(\varepsilon_{3.22} n - C_{3.22}))\, \mu^{\beta}_{\Lambda(l),\omega}(\Gamma^c_l)$$
$$\leqslant B_{3.43} \exp(-\beta l / C_{3.43})\, \mu^{\beta}_{\Lambda(l),\omega}(\Gamma^c_l). \tag{3.43}$$

Essentially the same argument, using $\mathscr{C}_-(\sigma, x)$ and S_- in place of $\mathscr{C}_+(\sigma, x)$ and S_+, shows that

$$\sum_{x \in \Lambda(l)} \sum_{\sigma \in \Gamma_l\colon \sigma^x \notin \Gamma_l, \sigma_x = -1} \mu^{\beta}_{\Lambda(l),\omega}(\sigma) \leqslant B_{3.44} \exp(-\beta l / C_{3.44})\, \mu^{\beta}_{\Lambda(l),\omega}(\Gamma^c_l). \tag{3.44}$$

We conclude (2.7) from (3.42), (3.43) and (3.44). ∎

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